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LINEAR ALGEBRA AND ITS APPLICATIONS

Linear rank and corank preserving maps on $\mathscr{B}(H)$ and an application to *-semigroup isomorphisms of operator ideals

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Abstract

We characterize linear rank-k nonincreasing, rank-k preserving, and corank-k preserving maps on $\mathscr{B}(H)$, the algebra of all bounded linear operators on the Hilbert space H. This unifies and extends finite-dimensional results and results on linear rank-1 nonincreasing and rank-1 preserving maps in the infinite-dimensional case. We conclude with an application to *-semigroup isomorphisms of operator ideals. © 1998 Elsevier Science Inc. All rights reserved.

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1. Introduction and statement of the results

One of the most frequently used methods to solve a particular linear preserver problem on a matrix algebra is to reduce this problem to that of characterizing linear rank preserving maps. It turns out that this method can be applied also in some infinite-dimensional linear preserver problems (see, for example [12,18,20]). So, it is not surprising that there is a vast literature on lin-

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ear maps preserving rank. Let us mention here just two recent finite-dimensional results: the result of Beasley [4] on linear rank-k preserving maps and the result of Loewy [13] on linear rank-k nonincreasing maps. Concerning infinite-dimensional operator algebras, only the special case $k = 1$ has been treated by now [11,17].

It is the aim of this note to characterize linear rank-k nonincreasing maps, linear rank-k preserving maps, and linear corank-k preserving maps on $\mathscr{B}(H)$, the algebra of all bounded linear operators on the Hilbert space H, thus unifying and extending the above mentioned results. The problem of corank-k preservers occurs, of course, in the infinite-dimensional case only. Here, it is also natural to ask about the structure of the set of all linear maps preserving both infinite rank and infinite corank. As we shall see later, the set of such maps is much larger than that of the previous ones.

Before stating our results we fix some notation. For an arbitrary pair of vectors x, y from a Hilbert space H we denote their scalar product by y^*x, while xy^* denotes the rank one operator defined by $(xy^*)z = (y^*z)x$. Note that every operator of rank one can be written in this form. Let U be a (not necessarily closed) linear subspace of H. We say that U is of finite codimension if $\dim U^{\perp} < \infty$. In this case we define the codimension of U by $\operatorname{codim} U = \dim U^{\perp}$. An operator $A \in \mathscr{B}(H)$ has finite corank if $\operatorname{Im} A$ is of finite codimension. In this case we define $\operatorname{corank} A = \operatorname{codim} \operatorname{Im} A$. Obviously, $\operatorname{corank} A = k$ if and only if $\dim \mathit{Ker}\, A^* = k$. For any positive integer k we denote by $\mathscr{B}_k(H), \mathscr{B}_{\leqslant k}(H)$ and $\mathscr{B}_{-k}(H)$ the set of all operators of rank k, rank at most k and corank k, respectively. We say that a linear map $\phi : \mathscr{B}(H) \to \mathscr{B}(H)$ is a rank-k preserver (a rank-k nonincreasing map) if $A \in \mathscr{B}_k(H)$ implies $\phi(A) \in \mathscr{B}_k(H)$ ($A \in \mathscr{B}_k(H)$ implies $\phi(A) \in \mathscr{B}_{\leqslant k}(H)$). Similarly, ϕ is said to preserve corank k in both directions provided that $A \in \mathscr{B}_{-k}(H)$ if and only if $\phi(A) \in \mathscr{B}_{-k}(H)$.

Our first result unifies and extends the result of Loewy [13] on linear maps on matrix algebras which are rank-k nonincreasing and the result of Hou [11] on rank-1 nonincreasing linear maps in the infinite-dimensional case. In our proof we will use both of these results.

Theorem 1. *Let k be a positive integer, and H be a Hilbert space. Assume that $\phi : \mathscr{B}(H) \to \mathscr{B}(H)$ is a linear rank-k nonincreasing map which is weakly continuous on norm bounded sets. Then either the image of ϕ is a linear space consisting of operators of rank at most k, or there exist $A, B \in \mathscr{B}(H)$ such that either $\phi(T) = ATB$ for all $T \in \mathscr{B}(H)$, or $\phi(T) = AT^{\mathrm{tr}}B$ for all $T \in \mathscr{B}(H)$, where T^{tr} denotes the transpose of T relative to any orthonormal basis of H fixed in advance.*

Note that in the above result the weak continuity assumption is essential. Namely, without this assumption nothing can be said about the behaviour of ϕ outside $\mathscr{F}(H)$, the ideal of all bounded linear finite rank operators. Of course, this assumption is automatically fulfilled in the finite-dimensional case.

Next, we will generalize the result of Beasley [4] on linear rank-k preservers on matrix algebras and the result of Hou [11] on linear rank-1 preservers in the infinite-dimensional case.

Theorem 2. *Let k be a positive integer, and H be a Hilbert space. Assume that $\phi : \mathscr{B}(H) \to \mathscr{B}(H)$ is a linear rank-k preserving map which is weakly continuous on norm bounded sets. Assume also that the image of ϕ is not contained in $\mathscr{B}_k(H)$. Then there exist an injective operator $A \in \mathscr{B}(H)$ and an operator $B \in \mathscr{B}(H)$ with dense image such that either $\phi(T) = ATB$ for all $T \in \mathscr{B}(H)$, or $\phi(T) = AT^{\mathrm{tr}}B$ for all $T \in \mathscr{B}(H)$.*

The finite-dimensional analogue of this result holds without the assumption that the image of ϕ contains an operator of rank greater than k (see [4]). However, this assumption is indispensable in the infinite-dimensional case. To see this let k be a positive integer and consider a separable infinite-dimensional Hilbert space with orthonormal basis $\{e_n\colon n = 1, 2, \ldots\}$. Define a family $T_n \in \mathscr{B}(H)$, $n = 1, 2, \ldots$, by

$$T_n e_j = \begin{cases} e_{j-n+1} & \text{if } n \leqslant j \leqslant n+k-1, \\ 0 & \text{otherwise.} \end{cases}$$

For every $A \in \mathscr{B}(H)$ order the countable set $\{e_j^* A e_i\colon i, j = 1, 2, \ldots\}$ into a sequence $(a_n)_{n=1}^{\infty}$. Use the same ordering for all operators and define $\phi : \mathscr{B}(H) \to \mathscr{B}(H)$ by

$$\phi(A) = \sum_{n=1}^{\infty} \frac{a_n}{n^2} T_n.$$

If A is nonzero then at least one a_n is nonzero, and hence $\phi(A)$ has rank k. Therefore, ϕ is linear rank-k preserving map weakly continuous on norm bounded sets, but is not of one of the forms described in the above theorem.

In the case of corank-k preserving maps we shall need stronger assumptions than in the case of rank-k preserving maps. Namely, we shall get our result under the stronger assumptions of bijectivity and preserving corank k in both directions.

Theorem 3. *Let k be a positive integer, and H be an infinite-dimensional Hilbert space. Assume that $\phi : \mathscr{B}(H) \to \mathscr{B}(H)$ is a linear bijective map weakly continuous on norm bounded sets which preserves corank-k operators in both directions. Then there exist invertible operators $A, B \in \mathscr{B}(H)$ such that $\phi(T) = ATB$ for all $T \in \mathscr{B}(H)$.*

We have already mentioned that many linear preserver problems were solved by reducing them to rank preserver problems. Here is another example.

Following Hestenes [10] we say that two operators $T, S \in \mathscr{B}(H)$ are orthogonal ($T \perp S$), if $T^*S = TS^* = 0$. In the following theorem we characterize additive maps preserving orthogonality.

Theorem 4. *Let H be a Hilbert space,* $\dim H > 1$, *and $\mathscr{A} \subset \mathscr{B}(H)$ be an ideal. Assume that $\phi : \mathscr{A} \to \mathscr{A}$ is an additive bijection which preserves orthogonality in both directions. Then there exist a nonzero constant c and unitary or antiunitary operators $U, V \in \mathscr{B}(H)$ such that either $\phi(T) = cUTV$ for all $T \in \mathscr{B}(H)$, or $\phi(T) = cUT^{\mathrm{tr}}V$ for all $T \in \mathscr{B}(H)$.*

This result is related to Theorem 2 of [15], where additive maps preserving a stronger orthogonality relation were considered.

In our final theorem we solve an open problem raised in [16] concerning *-identities on operator ideals. Let ϕ be a bijective function (no additivity or continuity is assumed) on an operator ideal fulfilling the n-variable *-identity

$$\phi(\tau_1(T_1)\tau_2(T_2)\ldots\tau_n(T_n)) = \tau_1(\phi(T_1))\tau_2(\phi(T_2))\ldots\tau_n(\phi(T_n))$$

for all T_js, where every τ_j is fixed and is either the identity or the adjoint operation. We prove that if at least one τ_j is the adjoint operation, then ϕ is an additive triple automorphism. An additive function ψ is called a triple homomorphism if it satisfies

$$\psi(TS^*R) = \psi(T)\psi(S)^*\psi(R)$$

for all T, S, R. Consequently, we obtain the quite surprising result that on operator ideals, the most general n-variable *-identity is that of the triple automorphisms. We think that this result is interesting even in the case of matrix algebras.

It should be mentioned that the concept of linear triple isomorphisms plays essential role in the theory of infinite-dimensional holomorphy as well as in the study of isometries of associative and Jordan operator algebras (see, for example, [6] and the references therein).

As for the "additivity" part of our theorem below, we remark that the problem of additivity of *-semigroup isomorphisms between operator algebras was raised by Saitô and Sakai and was treated in a series of papers by Hakeda (see [9] and the references therein). Semigroup isomorphisms of standard operator algebras were considered in [19].

Theorem 5. *Let H be a Hilbert space with* $\dim H > 1$, *and $\mathscr{A} \subset \mathscr{B}(H)$ be an ideal. Let $2 \leqslant n$ be an integer, and for every $1 \leqslant j \leqslant n$ let τ_j be either the identity or the adjoint operation on $\mathscr{A}$. Suppose that $\phi : \mathscr{A} \to \mathscr{A}$ is a bijective function satisfying the identity*

$$\phi(\tau_1(T_1)\tau_2(T_2)\ldots\tau_n(T_n)) = \tau_1(\phi(T_1))\tau_2(\phi(T_2))\ldots\tau_n(\phi(T_n))$$

for all $T_1, T_2, \ldots, T_n \in \mathscr{A}$.

If there is an index j such that τ_j is the adjoint operation, then ϕ is an additive triple automorphism of $\mathscr{A}$.

If τ_j is the identity for all j, then ϕ is equal to an additive ring automorphism of $\mathscr{A}$ multiplied by an $(n-1)$th root of unity.

Note that every operator ideal is self-adjoint (closed under taking adjoints) and hence the expression $\phi(\tau_1(T_1)\tau_2(T_2)\dots\tau_n(T_n))$ is well-defined even in the case when all τ_js are the adjoint operation.

2. Proofs

Proof of Theorem 1. Assume first that all finite rank operators from $\mathscr{B}(H)$ are mapped by ϕ into $\mathscr{B}_{\leqslant k}(H)$. It is easy to see that $\mathscr{B}_{\leqslant k}(H)$ is closed in the weak operator topology. Let T be any operator from $\mathscr{B}(H)$. Then we can find a bounded net of finite rank operators weakly converging to T. It follows that $\phi(T)$ belongs to $\mathscr{B}_{\leqslant k}(H)$. So, if there is an operator S of rank greater than k in the image of ϕ, then there is a finite rank operator R such that the rank of $\phi(R)$ is also greater than k. We will assume from now on that this is the case.

We shall show that ϕ is rank-1 nonincreasing. First note, that if P is any projection (self-adjoint idempotent) of finite rank p, then the algebra $P\mathscr{B}(H)P$ is isomorphic to M_p, the algebra of all $p \times p$ complex matrices. Assume that there is a rank one operator W such that $\phi(W)$ has rank greater than one. Then we can find two finite rank projections $P, Q \in \mathscr{B}(H)$ such that $PWP = W$, $PRP = R$, the rank of $Q\phi(W)Q$ is greater than one, and the rank of $Q\phi(R)Q$ is greater than k. By enlaring P or Q, if necessary, we can assume that the ranks of P and Q are the same, say p. Composing the natural isomorphism between M_p and $P\mathscr{B}(H)P$, as well as the one between $Q\mathscr{B}(H)Q$ and M_p, with our map ϕ in the following way

$$M_p \to P\mathscr{B}(H)P \xrightarrow{\phi} Q\mathscr{B}(H)Q \to M_p$$

we get a linear rank-k nonincreasing map from M_p into itself. The image of this map obviously contains a matrix with rank greater than k, so, by the theorem of Loewy [13] it is also rank-1 nonincreasing. This contradicts the fact that the rank of $\phi(W)$ is greater than one.

Hence, ϕ is rank-1 nonincreasing and we can apply a result of Hou (Corollary 1.1 of [11]) to complete the proof. Two minor remarks should be added here. Namely, when characterizing linear rank-1 nonincreasing maps Hou used the slightly stronger assumption that ϕ is weakly continuous on the whole $\mathscr{B}(H)$. It is easy to see that his proof works also under our assumption of weak continuity on norm bounded sets only. The other remark is that he formulated his result for general Banach spaces. So, he had to use the adjoint operator (here, the adjoint is meant in the Banach space sense) where we use the transpose.

Proof of Theorem 2. This is a direct consequence of Theorem 1.

Proof of Theorem 3. Our proof is based on the following characterization of rank one operators. A nonzero operator $T \in \mathscr{B}(H)$ has rank one if and only if for every corank-k operator S one of the following possibilities holds true: either $\alpha T + S \in \mathscr{B}_{-k}(H)$ for all but at most one complex number α, or $\alpha T + S \notin \mathscr{B}_{-k}(H)$ for all nonzero complex number α.

So, assume for a moment that we have already proved this characterization. Then, clearly, ϕ preserves operators of rank one. Applying Theorem 2, ϕ must be either of the form $\phi(T) = ATB$, or of the form $\phi(T) = AT^{\mathrm{tr}}B$ for some $A, B \in \mathscr{B}(H)$. Since ϕ is bijective, the operators A and B must be invertible. If T^{tr} denotes the transpose relative to the orthonormal basis $\{e_\alpha\colon\ \alpha \in J\}$, then $T^{\mathrm{tr}} = UT^*U$, where U is an antiunitary operator defined by

$$U\left(\sum_{\alpha\in J}\langle x, e_\alpha\rangle e_\alpha\right) = \sum_{\alpha\in J}\langle e_\alpha, x\rangle e_\alpha.$$

It is well known that in general dim Ker $T^* = k$ does not imply dim Ker $T = k$. So, the map $T \mapsto T^{\mathrm{tr}}$ does not preserve corank k, and hence, the second possibility cannot occur.

It remains to prove our characterization of rank one operators. Assume first that rank $T = 1$ and $S \in \mathscr{B}_{-k}(H)$. So, T is of the form $T = xy^*$ for some nonzero $x, y \in H$. Let W denote $S^{*^{-1}}(\mathrm{span}\{y\})$. The condition $\alpha T + S \in \mathscr{B}_{-k}(H)$ is equivalent to dim $Ker(\overline{\alpha}T^* + S^*) = k$. Because the kernel of $\overline{\alpha}T^* + S^*$ is contained in W, we have

$$\mathrm{Ker}(\overline{\alpha}T^* + S^*) = \mathrm{Ker}(\overline{\alpha}T^*_{|W} + S^*_{|W}).$$

Clearly, the restrictions $T^*_{|W}$ and $S^*_{|W}$ can be considered as linear maps from W to the linear span of y. Therefore, they can be represented as row matrices.

We have to distinguish two cases. Assume first that y belongs to the image of S^*. Then it follows from $S \in \mathscr{B}_{-k}(H)$ that dim $W = k + 1$. So, the restrictions $T^*_{|W}$ and $S^*_{|W}$ have matrix representations $[t_1, \ldots, t_{k+1}]$ and $[s_1, \ldots, s_{k+1}] \neq 0$, respectively. The operator $\alpha T^* + S^*$ does not belong to $\mathscr{B}_{-k}(H)$ if and only if $\alpha t_1 + s_1 = \cdots = \alpha t_{k+1} + s_{k+1} = 0$. This can happen for at most one complex number α.

In the remaining case that y does not belong to the image of S^* we have dim $W = k$ and $S^*_{|W} = 0$. In the case that $T^*_{|W} = 0$ we have $\alpha T + S \in \mathscr{B}_{-k}(H)$ for every complex number α, while $T^*_{|W} \neq 0$ implies that $\alpha T + S \notin \mathscr{B}_{-k}(H)$ for every nonzero α.

To prove the converse assume that rank $T > 1$. Once again we consider two cases. First, assume that dim $Ker\, T^* \geqslant k - 1$. Then we can find invertible operators $P, Q \in \mathscr{B}(H)$ such that T^* has the following matrix representation

$$PT^*Q = \begin{bmatrix} T_1 & T_2 \\ 0 & T_3 \end{bmatrix},$$

where T_1 is a $(k+1) \times (k+1)$-diagonal matrix $T_1 = \mathrm{Diag}(1, 1, 0, 0, \ldots, 0)$. Define $S \in \mathscr{B}_{-k}(H)$ by

$$PS^*Q = \begin{bmatrix} S_1 & 0 \\ 0 & \delta I \end{bmatrix},$$

where S_1 is a $(k+1) \times (k+1)$-diagonal matrix $S_1 = \mathrm{Diag}(1, 0, 0, \ldots, 0)$ and $|\delta| > ||T_3||$. It is easy to see that $-T + S$ and S belong to $\mathscr{B}_{-k}(H)$, while $\alpha T + S \notin \mathscr{B}_{-k}(H)$ for every real $\alpha \in (0, 1)$.

Now, if dim $Ker\, T^* < k - 1$, then we can find invertible operators $P, Q \in \mathscr{B}(H)$ such that T^* has the following matrix representation

$$PT^*Q = \begin{bmatrix} T_1 & T_2 \\ 0 & T_3 \end{bmatrix},$$

where T_1 is a $2k \times 2k$ identity matrix. Define $S \in \mathscr{B}_{-k}(H)$ by

$$PS^*Q = \begin{bmatrix} S_1 & 0 \\ 0 & \delta I \end{bmatrix},$$

where S_1 is a $2k \times 2k$-diagonal matrix having first k diagonal elements equal to zero and the rest of them equal to one. Let $|\delta| > ||T_3||$ be just as in the previous case. It is easy to see that $-T + S$ and S belong to $\mathscr{B}_{-k}(H)$, while $\alpha T + S \notin \mathscr{B}_{-k}(H)$ for every real $\alpha \in (0, 1)$. This completes the proof.

Proof of Theorem 4. Without further mentioning we will use the fact that every nontrivial ideal contains $\mathscr{F}(H)$.

We assert that ϕ is a rank-1 preserving map. To this end take any T of rank one and denote $S = \phi(T)$. Assume to the contrary that S has rank greater than one. If the spectrum of $|S|$ consists of one point only, then $|S|$ is a scalar multiple of the identity, and consequently, it can be written as the orthogonal sum of two nonzero positive operators $R_1, R_2 \in \mathscr{A}$. Using spectral theory we can decompose $|S|$ into the sum of two nonzero positive operators $R_1, R_2 \in \mathscr{A}$ also when the spectrum of $|S|$ is not a singleton. Note, that because of the ideal structure of $\mathscr{A}, R_1, R_2$ can be chosen to belong to $\mathscr{A}$. Now, let $R_3 = UR_1$ and $R_4 = UR_2$, where U is the partial isometry in the polar decomposition of S. Then, clearly $S = R_3 + R_4$ with $R_3, R_4 \in \mathscr{A}$ being orthogonal. Hence, $T = \phi^{-1}(S)$ is a sum of two orthogonal nonzero operators $\phi^{-1}(R_3)$ and $\phi^{-1}(R_4)$. Consequently, T has rank greater than one. This contradiction shows that ϕ preservers rank one operators. As ϕ preserves the orthogonality in both directions it must preserve also rank one operators in both directions. Hence, the restriction of ϕ to $\mathscr{F}(H)$ is a bijective additive map of $\mathscr{F}(H)$ onto itself preserving operators of rank one in both directions. It follows from Theorem 3.3 of [17] that there exist a ring automorphism h of the complex field and bijective h-quasilinear operators $A, B : H \to H$ such that either $\phi(xy^*) = (Ax)(By)^*$ for all $x, y \in H$ or $\phi(xy^*) = (Ay)(Bx)^*$ for all $x, y \in H$. Suppose that ϕ is of the first form above. We claim that h is either the identity or the conjugation. To see this, it is enough to show that h is a real-valued function on the real numbers. If r is a real number, then consider the rank-one operators $T = (e + rf)(e + 2f)^*$ and $S = (-2re + 2f)(2e - f)^*$, where $e, f \in H$ are orthogonal unit vectors. It is trivial to check that T and S are orthogonal. Consequently, $\phi(T)^*\phi(S) = 0$. Using the easy fact the A, B maps orthogonal vectors into orthogonal vectors, we obtain $-2h(r) + 2\overline{h(r)} = 0$. Without serious loss of generality, we may suppose that A, B are linear.

Since $\langle x, y\rangle = 0$ if and only if $\langle Ax, Ay\rangle = 0$, by the linearity of A we have $\langle Ax, Ay\rangle = c\langle x, y\rangle$, $x, y \in H$, for some complex constant c. So, both A and B are scalar multiples of unitary operators U and V^*. It follows that we have $\phi(T) = \lambda UTV$ for every finite rank operator T.

After multiplying ϕ from both sides by appropriate operators we can assume with no loss of generality that $\phi(T) = T$ for every finite rank operator T. It remains to show that this holds true for every $T \in \mathscr{A}$, too. As ϕ preserves orthogonality in both directions, we have $A \perp T + B$ if and only if $A \perp \phi(T) + B$ for every finite rank operators A and B. Take any $x \in H$. Let P be the projection onto the subspace generated by the vectors x, Tx, T^*x. Define $B = -PTP$. It is obvious that $B \in \mathscr{F}(H)$ and $Bx = -Tx$, $B^*x = -T^*x$. Let $A = xx^*$. Plainly, A is orthogonal to $T + B$, and consequently, it must be orthogonal to $\phi(T) + B$, which yields $Bx = -\phi(T)x$. As a consequence we have $\phi(T)x = Tx$. This completes the proof in the case which we have considered.

The remaining cases can be treated in a similar way.

Proof of Theorem 5. We first show that ϕ is additive. We use an argument similar to that in [14]. Let $T \in \mathscr{A}$ such that $\phi(T) = 0$. Then we have

$$\begin{aligned}\phi(0) &= \phi(\tau_1(T)\tau_2(0)\dots\tau_n(0))\\ &= \tau_1(\phi(T))\tau_2(\phi(0))\dots\tau_n(\phi(0)) = 0.\end{aligned}$$

It is not hard to see that we can assume that $n \geqslant 3$. In fact, if our equation is of the form

$$\phi(\tau_1(T)\tau_2(S)) = \tau_1(\phi(T))\tau_2(\phi(S)),$$

then write $T = ZW$ in the above expression and compute to get an equality in three variables. For example, if our equation is

$$\phi(TS^*) = \phi(T)\phi(S)^*,$$

then using the substitution $T = ZW$, we have

$$\phi(ZWS^*) = \phi(ZW)\phi(S)^* = \phi(Z)\phi(W^*)^*\phi(S)^*$$

which can be rewritten as

$$\phi(ZW^*S^*) = \phi(Z)\phi(W)^*\phi(S)^*.$$

So, let $n \geqslant 3$. We next assert that we can suppose that there is an index $1 < i < n$ for which $\tau_i(T) = T$ ($T \in \mathscr{A}$). Indeed, if for every $i = 1, \dots, n$ we have $\tau_i(T) = T^*$, then replacing T_1 by $Z_1 \dots Z_n$ in our equation, we arrive at

$$\begin{aligned}\phi((Z_1 \dots Z_n)^* T_2^* \dots T_n^*) &= \phi(Z_1 \dots Z_n)^* \phi(T_2)^* \dots \phi(T_n)^* \\ &= (\phi(Z_1^*)^* \dots \phi(Z_n^*)^*)^* \phi(T_2)^* \dots \phi(T_n)^* \\ &= \phi(Z_n^*) \dots \phi(Z_1^*) \phi(T_2)^* \dots \phi(T_n)^*,\end{aligned}$$

which yields

$$\phi(Z_n \dots Z_1 T_2^* \dots T_n^*) = \phi(Z_n) \dots \phi(Z_1) \phi(T_2)^* \dots \phi(T_n)^*$$

and this fulfills our requirements. One can follow the same argument if $\tau_1(T) = \dots = \tau_{n-1}(T) = T^*, \tau_n(T) = T$ or $\tau_1(T) = T$, $\tau_2(T) = \dots = \tau_{n-1}(T) = T^*, \tau_n(T) = T$. Finally, we replace T_n by $Z_1 \cdots Z_n$ if $\tau_1(T) = T$, $\tau_2(T) = \dots = \tau_n(T) = T^*$. From now on we assume that $n \geqslant 3$ and that there is an index $1 < i < n$ for which $\tau_i(T) = T$.

Let $T, S \in \mathscr{A}$ be fixed and E be a finite-rank projection. Pick arbitrary projections $P, Q \in \mathscr{A}$ of finite-rank. Since ϕ is a bijection, there is a unique $A \in \mathscr{A}$ for which $\phi(A) = \phi(TE) + \phi(S(I - E))$. We obtain

$$\begin{aligned}&\phi(P \dots PAQ \dots Q) \\ &\quad = \tau_1(\phi(\tau_1(P))) \dots \tau_{i-1}(\phi(\tau_{i-1}(P)))\phi(A)\tau_{i+1}(\phi(\tau_{i+1}(Q))) \dots \tau_n(\phi(\tau_n(Q))) \\ &\quad = \tau_1(\phi(\tau_1(P))) \dots \tau_{i-1}(\phi(\tau_{i-1}(P)))\phi(TE)\tau_{i+1}(\phi(\tau_{i+1}(Q))) \dots \tau_n(\phi(\tau_n(Q))) \\ &\qquad + \tau_1(\phi(\tau_1(P))) \dots \tau_{i-1}(\phi(\tau_{i-1}(P)))\phi(S(I - E)) \\ &\qquad \tau_{i+1}(\phi(\tau_{i+1}(Q))) \dots \tau_n(\phi(\tau_n(Q))) \\ &\quad = \phi(P \dots P(TE)Q \dots Q) + \phi(P \dots P(S(I - E))Q \dots Q).\end{aligned}$$

This implies that $PAQ = P(TE)Q + PS(I - E)Q$ if $Q = E$ or if $Q \perp E$. Since P was arbitrary, these result in $A = TE + S(I - E)$ and we have

$$\phi(TE + S(I - E)) = \phi(TE) + \phi(S(I - E)).$$

One can similarly verify that

$$\phi(ET + (I - E)S) = \phi(ET) + \phi((I - E)S).$$

Now, let $A \in \mathscr{A}$ be such that $\phi(A) = \phi(ET(I-E)) + \phi(ES(I-E))$. Let $\tilde{T} = ET(I-E)$ and $\tilde{S} = ES(I-E)$. We compute

$$\begin{aligned}
&\phi(P\ldots PAQ\ldots Q)\\
&= \tau_1(\phi(\tau_1(P)))\ldots\tau_{i-1}(\phi(\tau_{i-1}(P)))\phi(A)\tau_{i+1}(\phi(\tau_{i+1}(Q)))\ldots\tau_n(\phi(\tau_n(Q)))\\
&= \tau_1(\phi(\tau_1(P)))\ldots\tau_{i-1}(\phi(\tau_{i-1}(P)))\phi(ET(I-E))\\
&\quad \tau_{i+1}(\phi(\tau_{i+1}(Q)))\ldots\tau_n(\phi(\tau_n(Q)))\\
&\quad + \tau_1(\phi(\tau_1(P)))\ldots\tau_{i-1}(\phi(\tau_{i-1}(P)))\phi(ES(I-E))\\
&\quad \tau_{i+1}(\phi(\tau_{i+1}(Q)))\ldots\tau_n(\phi(\tau_n(Q)))\\
&= 0 + \phi(P(ET(I-E))Q) + \phi(P(ES(I-E))Q)\\
&= \tau_1(\phi(\tau_1(P)))\ldots\tau_{i-1}(\phi(\tau_{i-1}(P)))\phi(E)\tau_{i+1}(\phi(\tau_{i+1}((I-E)Q)))\\
&\quad \tau_{i+2}(\phi(\tau_{i+2}(Q)))\ldots\tau_n(\phi(\tau_n(Q)))\\
&\quad + \tau_1(\phi(\tau_1(P)))\ldots\tau_{i-1}(\phi(\tau_{i-1}(P)))\phi(\tilde{T})\tau_{i+1}(\phi(\tau_{i+1}((I-E)Q)))\\
&\quad \tau_{i+2}(\phi(\tau_{i+2}(Q)))\ldots\tau_n(\phi(\tau_n(Q)))\\
&\quad + \tau_1(\phi(\tau_1(P)))\ldots\tau_{i-1}(\phi(\tau_{i-1}(P)))\phi(E+\tilde{T})\\
&\quad \tau_{i+1}(\phi(\tau_{i+1}(\tilde{S}Q)))\tau_{i+2}(\phi(\tau_{i+2}(Q)))\ldots\tau_n(\phi(\tau_n(Q))) = (*).
\end{aligned}$$

Since by what we have proved above we know $\phi(E) + \phi(\tilde{T}) = \phi(E+\tilde{T})$, hence we have

$$\begin{aligned}
(*) &= \tau_1(\phi(\tau_1(P)))\ldots\tau_{i-1}(\phi(\tau_{i-1}(P)))\phi(E+\tilde{T})\tau_{i+1}(\phi(\tau_{i+1}((I-E)Q)))\\
&\quad \tau_{i+2}(\phi(\tau_{i+2}(Q)))\ldots\tau_n(\phi(\tau_n(Q))) + \tau_1(\phi(\tau_1(P)))\ldots\tau_{i-1}(\phi(\tau_{i-1}(P)))\\
&\quad \phi(E+\tilde{T})\tau_{i+1}(\phi(\tau_{i+1}(\tilde{S}Q)))\tau_{i+2}(\phi(\tau_{i+2}(Q)))\ldots\tau_n(\phi(\tau_n(Q))) = (**).
\end{aligned}$$

We also know that $\phi(\tau_{i+1}((I-E)Q)) + \phi(\tau_{i+1}(\tilde{S}Q)) = \phi(\tau_{i+1}((I-E)Q + \tilde{S}Q))$, hence we can continue

$$\begin{aligned}
(**) &= \tau_1(\phi(\tau_1(P)))\ldots\tau_{i-1}(\phi(\tau_{i-1}(P)))\phi(E+\tilde{T})\tau_{i+1}(\phi(\tau_{i+1}((I-E)Q\\
&\qquad + \tilde{S}Q)))\tau_{i+2}(\phi(\tau_{i+2}(Q)))\ldots\tau_n(\phi(\tau_n(Q)))\\
&= \phi(P\ldots P(E+\tilde{T})((I-E)Q+\tilde{S}Q)Q\ldots Q)\\
&= \phi(P(\tilde{T}+\tilde{S})Q).
\end{aligned}$$

By the injectivity of ϕ it follows that $PAQ = P(\tilde{T}+\tilde{S})Q$ for every projection $P, Q \in \mathscr{F}(H)$. Plainly, this implies that $A = \tilde{T} + \tilde{S}$ which yields $\phi(ET(I-E) + ES(I-E)) = \phi(ET(I-E)) + \phi(ES(I-E))$.

Now, let $A \in \mathscr{A}$ be such that $\phi(A) = \phi(ETE) + \phi(ESE)$. Moreover, let $\tilde{T} = ETE$ and $\tilde{S} = ESE$. Suppose that τ_1 is the identity on $\mathscr{A}$. Then we have

$$\begin{aligned}
&\phi(AQ(I-E)) \\
&\quad = \phi(A)\tau_2(\phi(\tau_2(Q)))\dots\tau_{n-1}(\phi(\tau_{n-1}(Q)))\tau_n(\phi(\tau_n(Q(I-E)))) \\
&\quad = \phi(\tilde{T})\tau_2(\phi(\tau_2(Q)))\dots\tau_{n-1}(\phi(\tau_{n-1}(Q)))\tau_n(\phi(\tau_n(Q(I-E)))) \\
&\qquad + \phi(\tilde{S})\tau_2(\phi(\tau_2(Q)))\dots\tau_{n-1}(\phi(\tau_{n-1}(Q)))\tau_n(\phi(\tau_n(Q(I-E)))) \\
&\quad = \phi(\tilde{T}Q(I-E)) + \phi(\tilde{S}Q(I-E)) = (***).
\end{aligned}$$

But from the previous step we obtain

$$\phi(\tilde{T}Q(I-E)) + \phi(\tilde{S}Q(I-E)) = \phi(\tilde{T}Q(I-E) + \tilde{S}Q(I-E))$$

and this implies

$$(***) = \phi(\tilde{T}Q(I-E) + \tilde{S}Q(I-E)) = \phi((\tilde{T}+\tilde{S})Q(I-E)).$$

By the injectivity of ϕ we have $AQ(I-E) = (\tilde{T}+\tilde{S})Q(I-E)$ for every Q. It is not hard to see that this gives $A = \tilde{T} + \tilde{S}$. If τ_1 is the adjoint operation, then one can argue in a similar way.

To prove the additivity, finally let $A \in \mathscr{A}$ be such that $\phi(A) = \phi(T) + \phi(S)$. Just as above we easily obtain $\phi(PAP) = \phi(PTP) + \phi(PSP)$. By what we already know, it follows $\phi(PTP) + \phi(PSP) = \phi(PTP + PSP) = \phi(P(T+S)P)$. Conseqently, $\phi(PAP) = \phi(P(T+S)P)$ and this implies $PAP = P(T+S)P$ for every finite rank projection P. Therefore, $A = T + S$ and this gives us $\phi(A+B) = \phi(A) + \phi(B)$.

Suppose that there exists an index i such that $\tau_i(T) = T^*$ for all $T \in \mathscr{A}$. We show that in this case ϕ preserves the orthogonality in both directions. If there are indices j, k such that

$$\tau_j(T) = T,\ \ \tau_{j+1}(T) = T^* \quad \text{and} \quad \tau_k(T) = T^*,\ \ \tau_{k+1}(T) = T,$$

this follows from our basic equation and the bijectivity of ϕ. If this is not the case, then there are indices j, k such that either

$$\tau_j(T) = T,\ \ \tau_{j+1}(T) = T^* \quad \text{or} \quad \tau_k(T) = T^*,\ \ \tau_{k+1}(T) = T$$

(see our assumption and its justification in the beginning of the proof). Without serious loss of generality we can assume that our *-identity is in the form

$$\phi(T_1 \dots T_j S_1^* \dots S_k^*) = \phi(T_1)\dots\phi(T_j)\phi(S_1)^*\dots\phi(S_k)^*,$$

where $k + j = n$. We compute

$$
\begin{aligned}
&\phi(T_1 \dots T_j S_1^* \dots S_{k-1}^* (Z_1 \dots Z_k W_1^* \dots W_j^*)) \\
&\quad = \phi(T_1) \dots \phi(T_j) \phi(S_1)^* \dots \phi(S_{k-1})^* \phi((Z_1 \dots Z_k W_1^* \dots W_j^*)^*)^* \\
&\quad = \phi(T_1) \dots \phi(T_j) \phi(S_1)^* \dots \phi(S_{k-1})^* \phi(W_j \dots W_1 Z_k^* \dots Z_1^*)^* \\
&\quad = \phi(T_1) \dots \phi(T_j) \phi(S_1)^* \dots \phi(S_{k-1})^* \phi(Z_1) \dots \phi(Z_k) \phi(W_1)^* \dots \phi(W_j)^*.
\end{aligned}
$$

Because of the form of this equality, we obtain the orthogonality preserving property of ϕ.

Now, apply Theorem 4 to have a form of ϕ. Since it satisfies a *-identity, one can easily check that $|c| = 1$ and that the possibility $\phi(T) = UT^{\mathrm{tr}}V$ cannot occur. This gives the assertion.

If every τ_j, $1 \leqslant j \leqslant n$, is the identity, then we can apply the main result in [5] on surjective n-Jordan homomorphisms of prime rings. In our particular case this says that there is an $(n-1)$th root of identity λ and a ring automorphism ψ of $\mathscr{A}$ such that $\phi = \lambda\psi$.

The proof of Theorem 5 is complete.

3. Remarks

Theorem 1 gives an almost complete characterization of linear rank-k non-increasing maps. To get the complete understanding of the structure of such maps, one must characterize maximal linear subspaces of $\mathscr{B}(H)$ consisting of operators of rank not greater than k. This problem seems to be difficult even in the finite-dimensional case and of interest in algebra in general [1–3,7,8].

As for the remaining case of linear maps preserving infinite rank as well as infinite corank we define a linear map $\phi : \mathscr{B}(H) \to \mathscr{B}(H)$ by $\phi(A) = A + \psi(A)$, where ψ is any linear map from $\mathscr{B}(H)$ into $\mathscr{F}(H)$ with norm strictly less than 1. Then ϕ is bijective and obviously preserves operators of infinite rank and infinite corank in both directions. This example shows that the set of such maps is much larger than the set of rank-k preservers.

There are several possibilities how to define corank. Our definition that corank $A = k$ if and only if $\dim\,(\mathrm{Im}\,A)^\perp = \dim\ \mathrm{Ker}\,A^* = k$ corresponds to column rank for matrices. Another possible definition, that is, corank $A = k$ if and only if $\dim(\mathrm{Im}\,A^*)^\perp = \dim\ \mathrm{Ker}\,A = k$ corresponds to row rank for matrices. These two definitions do not coincide in the infinite dimensional case. So, we have also the third possibility that corank A is equal to k if and only if $\dim\ \mathrm{Ker}\,A = \dim\ \mathrm{Ker}\,A^* = k$. Among all three definitions only the last one has the property that corank $A = k$ if and only if corank $A^* = k$. So, it is not surprising that the analogue of Theorem 3 corresponding to this definition reads as follows: let k be a positive integer, and H be an infinite-dimensional Hilbert space. Assume that $\phi : \mathscr{B}(H) \to \mathscr{B}(H)$ is a linear bijective map weakly continuous on norm bounded sets satisfying $\dim\ \mathrm{Ker}\,A = \dim\ \mathrm{Ker}\,A^* = k$ if and only if $\dim \operatorname{Ker} \phi(A) = \dim \operatorname{Ker}(\phi(A))^* = k$. Then there exist invertible operators $A, B \in \mathscr{B}(H)$ such that either $\phi(T) = ATB$ for all $T \in \mathscr{B}(H)$, or $\phi(T) = AT^{\mathrm{tr}}B$ for all $T \in \mathscr{B}(H)$. The proof of this statement is similar to the proof of Theorem 3. It is based on the following characterization of rank one operators among all nonzero operators from $\mathscr{B}(H)$: a nonzero $T \in \mathscr{B}(H)$ has rank one if and only if for every $S \in \mathscr{B}'_{-k}(H)$ we have either $\alpha T + S \in \mathscr{B}'_{-k}(H)$ for all complex α but at most two, or $\alpha T + S \notin \mathscr{B}'_{-k}(H)$ for all nonzero complex α. Here, of course, $\mathscr{B}'_{-k}(H)$ stands for the set of all operators from $\mathscr{B}(H)$ of corank k with respect to our last definition. As the idea of the proof is almost the same as in the proof of Theorem 3, we omit the details.

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